An exactly certified upper bound of 0.5604007071302298 for the tetrahedron density, and the structure that stops it

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Abstract

The Turán density $\pi(K_4^{(3)})$ of the tetrahedron $K_4^{(3)}$ has been conjectured since 1941 to equal $\frac{5}{9}$, and the best published upper bound has stood essentially unchanged since 2012. We give an exactly certified upper bound

$$\pi(K_4^{(3)}) \;\le\; \frac{951}{1697} \;=\; 0.5604007071302298,$$

verified by exact rational arithmetic with 0 violations across all 49,242 constraints of a level-6 colour-refined flag-algebra program. This improves the bound proved by Baber's own certificate, 0.5614656206270598, by 1.064913×10^{-3}, closing 18.019% of the gap between that value and $\frac{5}{9}$. The improvement comes from implementing the refinement Baber named but did not build in [1]—using the colours of the non-labelled vertices—and a control isolates the gain: exhausting the descent inside Baber's own cone buys at most 8×10^{-6}, while the single change to the refined cone buys 5.389×10^{-4}. We accompany the bound with a collection of exact structural results on the extremal family and the profile polytope, and with a precise account of why the method cannot reach $\frac{5}{9}$: because Turán's construction has density exactly $\frac{5}{9}$, every valid bound of this kind satisfies $\lambda_N \ge \frac{5}{9}$ at every level, so a proof of the conjecture at any finite level would require that level to be *exactly* tight. The obstruction is exact tightness, not accuracy and not computational cost.

1 Introduction

For a 3-uniform hypergraph F, the Turán density $\pi(F)$ is the limiting maximum edge density of an n-vertex 3-graph containing no copy of F. The oldest open problem in the subject is Turán's 1941 conjecture that

$$\pi(K_4^{(3)}) = \tfrac{5}{9},$$

where $K_4^{(3)}$ is the complete 3-graph on four vertices (the tetrahedron). The lower bound $\pi(K_4^{(3)}) \ge \frac{5}{9}$ is realised by Turán's own construction: partition the vertices into three parts V_0, V_1, V_2 and take as edges every transversal triple together with every triple having two vertices in V_i and one in V_{i+1} (indices mod 3). The upper bound is the entire difficulty, and it is where all subsequent work has concentrated.

#	bound	exact value	% of gap closed
1	0.5609267206135859	280463360306793/500000000000000	9.118
2	0.5607731206133190	560773120613319/1000000000000000	11.717
3	0.5605389560458980	560538956045898/1000000000000000	15.679
4	0.5605204340454750	22420817361819/40000000000000	15.993
5	0.5604725127771610	560472512777161/1000000000000000	16.804
6	0.5604543821485490	560454382148549/1000000000000000	17.110
7	0.5604502462336350	112090049246727/200000000000000	17.180
8	0.5604277291727250	22417109166909/40000000000000	17.561
9	0.5604266698930960	70053333736637/125000000000000	17.579
10	0.5604253953924430	560425395392443/1000000000000000	17.601
11	0.5604244258925900	56042442589259/100000000000000	17.617
12	0.5604129143306370	560412914330637/1000000000000000	17.812
13	0.5604099228526640	70051240356583/125000000000000	17.863
14	0.5604086444902741	280204322245137/500000000000000	17.884
15	0.5604075691411936	385/687	17.903
16	0.5604066985645934	937/1672	17.917
17	0.5604060913705584	552/985	17.928
18	0.5604031126419187	4393/7839	17.978
19	**0.5604007071302298**	**951/1697**	**18.019**

Table 1: The certified bounds, strictly decreasing, each with 0 violations of the 49,242 constraints. Percentages are of the gap from Baber's certified value 0.5614656206270598 to $\frac{5}{9}$. Sources are the certification logs.

The record and the citation record. The flag-algebra method [13] reduced the upper bound to a semidefinite computation, and the resulting numbers have been quoted inconsistently for over a decade. Three values circulate. The figure 0.561666 that appears in most surveys and problem lists is a plain level-6 floating-point value that was never certified by its author; Keevash's survey [5] describes it as likely to remain unrigorous. Baber's published figure [1] is 0.5615. Baber's own certificate, however, proves the strictly stronger 0.5614656206270598, and this—not the published rounding, and not the uncertified float—is the correct incumbent against which any improvement should be measured. That [1] carries no journal reference helps explain how the citation drifted. We recover Baber's true value by re-checking his data, and it is the honest baseline for this paper.

Results. Our main result is the certified bound stated in the abstract. It is the last of a strictly decreasing sequence of nineteen certified bounds, listed in Table 1; every one lies below Baber's certified value, and each is verified in exact rational arithmetic against all 49,242 constraints.

Alongside the bound we prove a body of structural results, summarised here and proved in the indicated sections:

- *Forced flatness* (§3): in any 3-graph carrying Turán's cross-structure, freeness from $K_4^{(3)}$ forces every part to have no internal edge; the extremal family is therefore not a naive iteration.
- A *martingale identity* for the local profile under vertex deletion (§4), from which every finite-level profile inequality is false at every finite n.
- A *plateau theorem* (§4): the counting certificate attains the Katona–Nemetz–Simonovits

bound [11] with exact equality precisely at $n \equiv 0 \pmod 3$, so one third of levels are dead weight.

- The first computation of the exact 4-profile polytope at $n = 7$ (§4): 15 vertices and 17 facets, all integer-verified.
- An account of *what the method cannot do* (§5): the exact-tightness obstruction, the tightness criterion for external inequalities, and the certified death of the family-only certificates at levels 6 and 7.

The honest limit, stated up front. Because Turán's construction has density exactly $\frac{5}{9}$, every valid upper bound produced by this method satisfies $\lambda_N \geq \frac{5}{9}$ at every level N. Consequently a proof of $\pi(K_4^{(3)}) \leq \frac{5}{9}$ at any finite level would require that level to be *exactly* tight: the residual $74/15273 = 4.845152 \times 10^{-3}$ would have to vanish identically, not merely shrink. In particular the statement "for every $\varepsilon > 0$ some finite level certifies $\pi(K_4^{(3)}) \leq \frac{5}{9} + \varepsilon$" is, since $\pi(K_4^{(3)}) \geq \frac{5}{9}$ is known, logically equivalent to the conjecture itself; it is not a weaker milestone, and a disproof of it in that form would disprove Turán. Our bound is the endpoint of a local descent in the refined cone, not the optimum of that program, and the method's own ceiling remains unbracketed from below within the interval that matters. None of this weakens the certified bound, which needs no optimality; but it is why the paper is a bound-and-structure paper and not a proof of the conjecture.

We use p_i throughout for the density of 4-vertex subsets spanning exactly i of their four triples; freeness from $K_4^{(3)}$ is exactly $p_4 = 0$, and d denotes edge density.

2 The colour-refined cone

We work with the level-6 flag-algebra program in the coloured theory of Baber [1, §5.1], augmented with the Hladký–Král–Norine regularity rows. The program has 49,242 constraint classes. Writing $\mathrm{num}_H(Q)$ for the flag-positivity contribution of a positive-semidefinite block matrix Q on class H and $\mathrm{den}_H(\mu)$ for the regularity denominator with multipliers $\mu \geq 0$, the certified quantity is

$$\rho^* \;=\; \max_H \frac{\mathrm{num}_H(Q)}{\mathrm{den}_H(\mu)}.$$

Because the regularity multipliers enter bilinearly, this is not a plain semidefinite program in (Q, μ); we solve it by an alternating descent whose inner step is a semidefinite solve at fixed μ. The descent algorithm applies to any flag-algebra problem carrying regularity constraints, independently of the tetrahedron.

Baber's improvement (1). Baber writes in [1, §5.1]: *"due to time restrictions we did not make full use of the information contained in the flags. In particular we chose to ignore the colours of the non-labelled vertices."* Building that refinement— carrying the colour of every non-labelled vertex rather than only the labelled ones—is the source of our improvement. The refinement enlarges the cone, and the enlargement is safe in the following exact sense.

Lemma 1 (Containment)**.** *Let M_H^{grey} be the block matrices of Baber's cone and M_H^{full} those of the colour-refined cone, and let Π be the projection collapsing non-labelled colours. Then $M_H^{\mathrm{grey}} = \Pi\, M_H^{\mathrm{full}}\, \Pi^{\mathsf{T}}$, and consequently for any $Q^{\mathrm{grey}} \succeq 0$ the matrix $Q^{\mathrm{full}} := \Pi^{\mathsf{T}} Q^{\mathrm{grey}} \Pi$ is positive semidefinite and feasible in the refined program. Hence the refined cone contains the grey cone, and any grey certificate is a feasible warm start.*

move	gain
descent to exhaustion inside Baber's grey cone (two runs)	1.0×10^{-6} to 7.7×10^{-6}
the colour refinement alone (grey $\to$ full, descent fixed)	5.389×10^{-4}
subsequent descent inside the refined cone	5.260×10^{-4}

Table 2: Isolating the source of the improvement. The refinement, not further descent in the old cone, carries the gain.

Proof. The colour-forgetting map Π is a linear surjection on the flag spaces intertwining the two families of block matrices by construction of the refined theory, which gives the stated identity. For $Q^{\text{grey}} \succeq 0$ and any vector x, $x^\mathsf{T} Q^{\text{full}} x = (\Pi x)^\mathsf{T} Q^{\text{grey}} (\Pi x) \geq 0$, so $Q^{\text{full}} \succeq 0$; feasibility of the induced certificate follows from the intertwining identity applied constraint by constraint. $\square$

The control. Lemma 1 lets us separate two questions that are otherwise confounded: how much room remains in Baber's own cone, and how much the refinement adds. Table 2 answers both. Descending to exhaustion inside the grey cone—two runs from different starts—moves the bound by at most 8×10^{-6}. The single change to the refined cone, holding the descent fixed, moves it by 5.389×10^{-4}, from Baber's 0.5614656206270598 to entry 1 of Table 1, 0.5609267206135859, a factor of about seventy. Continued descent inside the refined cone then carries the bound a further 5.260×10^{-4} to the record. The reading is unambiguous: Baber's certificate was already near-optimal *for his own cone*, and the room that remained in the level-6 coloured approach lay in the refinement he named and skipped.

Exact certification. A floating-point optimum is a measurement, not a proof. To certify a rational ρ we never round Q itself; we round its Cholesky factor to integers and set $\widehat{Q} := \widehat{R}^\mathsf{T} \widehat{R}/2^{2K}$, which is positive semidefinite by construction for any integer $\widehat{R}$, so no eigenvalue argument is needed. (This device is standard; Flagmatic and the FlagAlgebraToolbox both round to exact rationals, and we claim no priority for it.) With every regularity denominator cleared to an integer, each of the 49,242 constraint values becomes an exact rational whose sign is decidable. For the bound of Table 1 this returns 0 violations, with worst constraint value -1.430×10^{-7}, in about 56 seconds.

The certified value is reproducible under a change of the certification route. Factoring each block instead by an LDL^T (Bunch–Kaufman) decomposition at a different integer scale produces a different integer factor $\widehat{R}$ and a different worst constraint value, -1.392×10^{-7}, while returning the same 0 violations and the same certified rational; the two integer certificates differ (their SHA-256 digests are `26c689f4`... and `e40ce355`...). The bound therefore does not depend on the particular rounding.

Backing off from a float. The descent that produced entry 12 of Table 1 ended at the floating-point value 0.5604128143306374, with solver status `optimal_inaccurate` and a block of minimum eigenvalue -3.17×10^{-8}: not positive semidefinite, and hence not a bound. Exact certification backs off to 0.5604129143306370, higher by 10^{-10} and still a valid upper bound, where the rounded integer Cholesky makes positive-semidefiniteness structural. The float and the certificate are different numbers, and the certified one is what survived the exact check rather than a value chosen in advance. We report bounds only after this step.

A falsifier that fires. As a calibration we run the same certifier at the conjectured value $\frac{5}{9}$, where a correct bound must fail. Baber's certificate yields 3,083 violations of the 49,242 constraints and ours yields 4,516. Two things are worth noting. First, that the certifier does fire at $\frac{5}{9}$ is direct evidence the pipeline can detect an invalid bound. Second, and against intuition, our *better* bound has *more* violating constraints at $\frac{5}{9}$ than Baber's; the count of violations at a target below the true value carries no information about the quality of a bound, and here it runs opposite to it. We record the calibration because it is cheap, decisive, and rarely published.

Data. The certificate is released as data, in the manner of Baber's `K4.txt`. It was precisely the availability of his certificate as data that made his true value recoverable and this improvement possible.

3 Structure of the extremal family

The results of this section concern the *plain* program and the combinatorics of the extremal objects; they are not statements about the colour-refined cone in which the bound of §2 lives, and we keep the two separate.

Theorem 2 (Forced flatness)**.** *In any* 3*-graph containing Turán's cross-structure, freeness from* $K_4^{(3)}$ *forces every part to contain no internal edge.*

Proof. Take $x, y, z \in V_a$ and $w \in V_{a+1}$. Each of the triples $\{x, y, w\}$, $\{x, z, w\}$, $\{y, z, w\}$ has two vertices in V_a and one in V_{a+1}, hence is an edge unconditionally. The four-set $\{x, y, z, w\}$ therefore spans at least three edges, and spans four—a copy of $K_4^{(3)}$—if and only if the internal triple $\{x, y, z\}$ is an edge. Freeness excludes the latter, so no part carries an internal edge. □

Theorem 2 has three consequences. No recursion inside a part is possible; the extremal family is therefore *not* the naive iterated construction (a control confirms this: naive depth-2 recursion produces $p_4 = \frac{20}{243}$, an explicit tetrahedron); and the hand-recursion device that carried the Bodnár–León–Liu–Pikhurko proof for $C_5^{(3)-}$ [10] is unavailable here.

Theorem 3 (Orientation carries no information)**.** *The predecessor rule is the successor rule composed with the transposition of two parts, so the two constructions are isomorphic; the subgroup of* S_3 *preserving the successor rule is cyclic of order* 3 *and index* 2*.*

Proof. Composing the successor pairing $(0, 1), (1, 2), (2, 0)$ with the transposition $(V_1\, V_2)$ yields $(0, 2), (2, 1), (1, 0)$, the predecessor pairing, while transversal triples are fixed by any relabelling of parts; the two edge sets are thus related by a relabelling. A part-permutation preserves the successor rule if and only if it is a rotation, and the rotations form $C_3 \leq S_3$. □

Remark 4. Any argument that turns on the cyclic orientation of Turán's construction is therefore void.

Theorem 5 (Profile identity)**.** *For every* 3*-graph,* $3p_0 + 2p_1 + p_2 - p_4 = 3 - 4d$*. On* $K_4^{(3)}$*-free objects, where* $p_4 = 0$*, this reduces to* $3p_0 + 2p_1 + p_2 = 3 - 4d$*, and hence* $\pi(K_4^{(3)}) \leq \frac{5}{9}$ *if and only if* $3p_0 + 2p_1 + p_2 \geq \frac{7}{9}$*, a bound Turán's construction attains with equality.*

Proof. Counting, over a uniform random 4-set, $3 \sum_i p_i - \sum_i i\, p_i = 3 - 4d$ since $\sum_i i\, p_i = 4d$; expanding gives the stated identity. The p_4 term is genuine: on the binomial random 3-graph $G(n, \frac{5}{9})$ the left side equals $\frac{5728}{6561} = \frac{7}{9} + \left(\frac{5}{9}\right)^4$, not $\frac{7}{9}$. It is invisible precisely on $K_4^{(3)}$-free objects, which is the only place it may be dropped. □

The extremal family. The extremal objects form the Fon-der-Flaass–Brown–Kostochka family [8, 7, 6], a one-parameter curve in profile space,

$$\left(p_0,\ \tfrac{5}{27}-p_0,\ \tfrac{11}{27}-p_0,\ \tfrac{11}{27}+p_0,\ 0\right), \qquad p_0=\tfrac{4}{27}+\tfrac{1}{27k^2},$$

with exact members

$$k{=}1:\left(\tfrac{5}{27},0,\tfrac{6}{27},\tfrac{16}{27},0\right), \quad k{=}2:\left(\tfrac{17}{108},\tfrac{1}{36},\tfrac{1}{4},\tfrac{61}{108},0\right), \quad k{=}3:\left(\tfrac{37}{243},\tfrac{8}{243},\tfrac{62}{243},\tfrac{136}{243},0\right),$$

and limit point $\left(\frac{4}{27},\frac{1}{27},\frac{7}{27},\frac{15}{27},0\right)$ as $k\to\infty$. Every member satisfies $d=\frac{5}{9}$, $3p_0+2p_1+p_2=\frac{7}{9}$, and $p_0+p_1=\frac{5}{27}$ identically, and $p_1(k)=\frac{1}{27}(1-k^{-2})$, so $\sup_k p_1=\frac{1}{27}$, approached but never attained. The base construction ($k=1$) has the exact finite-n profile, at $n=3m$,

$$p_0=\frac{(m-2)(5m-3)}{3(3m-1)(3m-2)}, \quad p_1=0, \quad p_2=\frac{2m(m-1)}{(3m-1)(3m-2)}, \quad p_3=\frac{8m(2m-1)}{3(3m-1)(3m-2)},$$

so $p_1=0$ at every finite n, not only in the limit. At 5-vertex resolution the limit charges exactly 5 of the 23 $K_4^{(3)}$-free types, and its edge-count marginal is supported only on $\{0,3,6,7\}$; marginalising back to 4 vertices returns $\left(\frac{5}{27},0,\frac{6}{27},\frac{16}{27},0\right)$ exactly. An exhaustive census of the six-part construction at $d=\frac{5}{9}$ finds exactly nine digon-free digraphs, of which eight give the $k=2$ profile.

Theorem 6 (The Lagrangian, globally)**.** *For the Turán pattern the Lagrangian is* $L(a,b,c)=abc+\frac{1}{2}(a^2b+b^2c+c^2a)$ *and the blow-up density is* $d=6L$*. Its maximum on the simplex is* $\frac{5}{54}$ *at* $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$*, so* $6\lambda=\frac{5}{9}$*. The stationary ideal is zero-dimensional; the eliminant in the Lagrange multiplier is exactly* $(3\mu-5)(7\mu-9)$*, with a single real critical point in the simplex, and the boundary maxima are all* $\frac{4}{9}$*. In orthonormal tangent coordinates at the optimum,*

$$d=\tfrac{5}{9}-r^2+C(\theta)\,r^3$$

exactly, the r^0*,* r^1*, and* r^4 *coefficients vanishing identically.*

The isotropy of the quadratic term is not an independent finding: it is forced, because the two-dimensional real representation of C_3 on the tangent plane is irreducible, so every C_3-invariant quadratic form there is a scalar multiple of the identity.

The bound $d\le\frac{5}{9}$ on blow-ups also admits a fully explicit sum-of-squares certificate, which we record because it is elementary and self-checking:

$$\tfrac{5}{9}(a+b+c)^3-6abc-3(a^2b+b^2c+c^2a) \;=\; \sum_{\text{cyc}} a\left[\tfrac{5}{9}(a-c)^2-\tfrac{14}{9}(a-c)(b-c)+\tfrac{11}{9}(b-c)^2\right], \quad (1)$$

an identity in $\mathbb{Q}[a,b,c]$. Each bracket is the positive-semidefinite form $\frac{1}{9}\left(\begin{smallmatrix}5&-7&2\\-7&11&-4\\2&-4&2\end{smallmatrix}\right)$, of rank 2 with kernel $(1,1,1)$, and equals $\frac{5}{9}\left(u-\frac{7}{5}w\right)^2+\frac{2}{15}w^2$ with $u=a-c,\ w=b-c$. On the simplex $a,b,c\ge 0$, so the right side of (1) is a nonnegative combination of squares and $d\le\frac{5}{9}$; equality in the interior forces each rank-2 bracket to vanish, hence $a=b=c$, so the only interior maximiser is the centre. This reproves Theorem 6 for blow-ups without the Gröbner computation. We emphasise that (1) bounds the density of *blow-ups* of the pattern; it is not a bound on $\pi(K_4^{(3)})$, since a $K_4^{(3)}$-free 3-graph need not be a blow-up.

4 The counting certificate and the profile polytope

Theorem 7 (Monotonicity)**.** *The ratio* $\mathrm{ex}(n)/\binom{n}{3}$ *is non-increasing in* n*, and the proof uses only closure under vertex deletion.*

Proof. Let H be extremal on n vertices. Each edge lies in exactly $n-3$ of the n subsets of size $n-1$, and each such subset induces a $K_4^{(3)}$-free graph, so $(n-3)\,\mathrm{ex}(n) = \sum_S e(H[S]) \le n\,\mathrm{ex}(n-1)$. Since $\binom{n}{3}/\binom{n-1}{3} = n/(n-3)$ exactly, division gives $\mathrm{ex}(n)/\binom{n}{3} \le \mathrm{ex}(n-1)/\binom{n-1}{3}$. □

The argument needs closure under *vertex* deletion and not under edge deletion; the property should be called *hereditary*, not *monotone*, the latter meaning edge-deletion-closed in hypergraph usage. This is not pedantry: run the same averaging on the non-hereditary property "H has an even number of edges" and the inequality is violated at $n = 8$ ($5 \cdot 56 = 280 > 8 \cdot 34 = 272$) and again at $n = 12$, with the ratio strictly increasing.

Theorem 8 (Plateau)**.** *Write* $\mathrm{exc}(n)$ *for the excess of Turán's construction density over* $\frac{5}{9}$*. Then*

$$\mathrm{exc}(n) = \begin{cases} \dfrac{6n-10}{9(n-1)(n-2)}, & n \equiv 0, \\ \dfrac{6n-10}{9n(n-2)}, & n \equiv 1, \quad \pmod 3, \\ \dfrac{6n-4}{9n(n-1)}, & n \equiv 2, \end{cases}$$

and $\mathrm{exc}(3m+2) = \mathrm{exc}(3m+3) = \dfrac{18m+8}{9(3m+2)(3m+1)}$ *for every* m*. Hence Turán's construction attains the bound of Theorem 7 with exact equality precisely at* $n \equiv 0 \pmod 3$*.*

Proof. The three closed forms are the density of the balanced construction minus $\frac{5}{9}$, evaluated by residue and verified against direct edge counts for $n = 4, \ldots, 40$ with no discrepancy. Substituting $n = 3m+2$ and $n = 3m+3$ into the $\equiv 2$ and $\equiv 0$ forms gives numerator $18m+8$ and denominator $9(3m+2)(3m+1)$ in both cases. □

The plateau is visible in the known data: the flat steps of $\mathrm{ex}(n)/\binom{n}{3}$ occur at $(5,6), (8,9), (11,12), (14,15), (17,18)$ and nowhere else through $n = 18$—every one a $(3m+2, 3m+3)$ pair. (The values $\mathrm{ex}(n)$ are $3, 7, 14, 23, 36, 54$ for $n = 4, \ldots, 9$, established here by two solvers and a satisfiability proof, and 75, 102, 136, 174, 220, 275, 335, 405, 486 for $n = 10, \ldots, 18$; the latter are *proved* only for $n \le 13$, by Frohmader [4], who writes "verified for the cases $n \le 13$ by Spencer," with $n = 14, \ldots, 18$ resting on an unrefereed enumeration archive.)

The plateau also sharpens the comparison against the trivial counting certificate, $\mathrm{ex}(n)/\binom{n}{3}$, which converges to $\pi(K_4^{(3)})$ from above. To reach our own residual, that certificate needs level $n \ge 139$ (at $n = 138$ its excess is 4.878107×10^{-3}, still short; all three balanced assignments at $n = 139$ give the identical exact density); to reach Baber's residual it needs $n \approx 115$–117; and at the same level 6 it gives only $\frac{7}{10}$ against our 0.5604007071302298, an excess ratio of 29.81. One third of the levels between, moreover, are dead weight. This is the honest scale of what the semidefinite machinery buys: not an improvement over nothing, but an improvement over the dumbest certificate that provably converges.

Theorem 9 (Deletion martingale)**.** *For every* 3*-graph* H *on* n *vertices,* $p_i(H) = \frac{1}{n}\sum_v p_i(H - v)$*. Consequently the minimum of* $p_0 + p_1$ *over* n*-vertex* $K_4^{(3)}$*-free graphs increases to its limiting value from below, and any strict profile inequality for limit objects is false at every finite* n*.*

Proof. Each 4-set of H lies in exactly $n-4$ of the n vertex-deleted subgraphs, and $\binom{n-1}{4}n = \binom{n}{4}(n-4)$ identically, so averaging p_i over deletions recovers $p_i(H)$. Since each $H-v$ is again $K_4^{(3)}$-free, the minimum profile functional is a deletion-average of admissible values and so cannot lie below the limiting minimum. □

Corollary 10 (Nesting)**.** *The profile bodies are nested,* $\mathrm{conv}(P_4) \supseteq \mathrm{conv}(P_5) \supseteq \cdots$*, and every facet of every* $\mathrm{conv}(P_n)$ *is a valid inequality for all* $K_4^{(3)}$*-free limit objects.*

Proof. By Theorem 9 each n-profile is a convex combination of $(n-1)$-profiles, so $P_n \subseteq \mathrm{conv}(P_{n-1})$ and the bodies decrease; the limit body lies in every $\mathrm{conv}(P_n)$, so a valid inequality for the latter is valid for it. □

The $n = 7$ polytope. We computed the exact 4-profile polytope at $n = 7$. The 1,295,600 $K_4^{(3)}$-free classes realise 1,185 distinct 4-profiles, which—since $\sum p_i = 1$ and $p_4 = 0$—lie in three dimensions and form a polytope with 15 vertices and 17 facets. Turán's construction $(2, 0, 7, 26)/35$ is a vertex, with 23 edges, equal to $\mathrm{ex}(7)$. The thirteen non-trivial facets, written in integer counts out of $\binom{7}{4} = 35$ with c_3 eliminated, are

$$\begin{array}{lll}
-9c_0 - 6c_1 + 5c_2 \le 105, & -9c_0 + 2c_1 - 3c_2 \le 21, & -7c_0 - 6c_1 - c_2 \le -21, \\
-7c_0 - 2c_1 - 13c_2 \le -105, & -7c_0 - 2c_1 - 5c_2 \le -49, & -5c_0 + 4c_1 + 9c_2 \le 245, \\
-3c_0 + c_2 \le 21, & -3c_0 + c_1 + 3c_2 \le 77, & -3c_0 + 2c_1 - c_2 \le 39, \\
-c_0 - c_2 \le -7, & -c_0 + c_2 \le 25, & -c_0 + 4c_1 + c_2 \le 105, \\
3c_0 + 6c_1 + 5c_2 \le 189. & &
\end{array}$$

Every facet was verified tight and valid over all 1,185 profiles in exact integer arithmetic. By Corollary 10 each is a valid inequality for all limit objects.

Remark 11. The facets must be computed from the integer vertices, not by rationalising a floating-point hull. A first pass that rounded the floating-point facet equations of a convex-hull routine produced eight spurious "facets" out of seventeen, one of them violated by all 1,185 profiles.

One valid inequality is worth recording as a shadow of the polytope: for all $K_4^{(3)}$-free limit objects, $2d + (p_0 + p_1) \le \frac{7}{5}$, equivalently $p_0 + p_1 \ge \frac{11}{35} \Rightarrow d \le \frac{19}{35}$, tight at $n = 7$ on the edge between $(0, 7, 7, 21)$ and $(0, 21, 7, 7)$. It is *not* tight on the extremal family, which has $p_0 + p_1 = \frac{5}{27} < \frac{11}{35}$ and so violates its hypothesis.

Why the polytope route is closed. Three exact facts close the profile-geometry approach to $\frac{5}{9}$, and a reader who sees the facet table above will otherwise attempt exactly these routes. First, no facet is tight on the extremal family: across $n = 5, 6, 7$ the smallest slack of any facet on the family is *identically* $\frac{28}{135}$, achieved on the same facet $p_0 + p_2 \ge \frac{1}{5}$, because $p_0 + p_2$ is constant on the family at $\frac{11}{27}$ and a family-constant functional's slack cannot move with n. Second, that facet direction decomposes as $(1, 0, 1, 0) = 3 \cdot \mathbf{1} - 4d - 2(p_0 + p_1 - \frac{5}{27})$, so it lies in the two-dimensional space of family-constant functionals already spanned by the density and the auxiliary functional of §5; the hull produces nothing outside it. Third, the hull's density direction is the counting bound identically: the minimum of $3p_0 + 2p_1 + p_2$ over the $n = 7$ hull is $\frac{13}{35} = 3 - 4\,\mathrm{ex}(7)/\binom{7}{3}$, so "compute the hull at larger n and watch its minimum rise to $\frac{7}{9}$" is the counting certificate of Theorem 7 restated, needing $n \ge 139$.

5 What this method cannot do

Proposition 12 (Exact tightness is the obstruction)**.** *Turán's construction has density exactly* $\frac{5}{9}$*, so* $\pi(K_4^{(3)}) \geq \frac{5}{9}$ *and every valid upper bound* λ_N *of flag-algebra type satisfies* $\lambda_N \geq \frac{5}{9}$ *at every level* N*. Hence a proof of* $\pi(K_4^{(3)}) \leq \frac{5}{9}$ *at level* N *requires* $\lambda_N = \frac{5}{9}$ *exactly.*

The residual $74/15273 = 4.845152 \times 10^{-3}$ must therefore vanish identically at some finite level, not merely shrink; the obstruction is exact tightness, not accuracy and not cost. As noted in the introduction, this also makes "possibility" in the ε-sense equivalent to the conjecture, and its negation a disproof of Turán.

The natural way to push a flag bound below its value is to adjoin an external inequality, and there is an exact criterion for when this can help.

Theorem 13 (Tightness criterion)**.** *For an external functional* $\Phi \geq 0$ *adjoined with multiplier* $\mu \geq 0$*,* $\lambda(\mu) \geq \pi + \mu \cdot \max_{family} \mathrm{slack}(\Phi)$*. Consequently* Φ *can lower the bound only if it is tight on every member of the extremal family.*

Proof. Flag positivity gives $\lambda(\mu) \geq \pi + \mu\, \Phi(y)$ for every extremal y; taking the least favourable extremal point replaces $\Phi(y)$ by the maximal family slack. □

The worked example is the auxiliary inequality $p_0+p_1 \geq \frac{5}{27}$. It is tight on the whole family, is not derivable at level 6 (falling short there by $8.822832720422634 \times 10^{-3}$), and is logically independent of Turán's conjecture in both directions (by exact linear programming, $\min(p_0+p_1)$ subject to $d \leq \frac{5}{9}$ is 0, attained at $(0, 0, 1, 0)$, and $\min(3p_0+2p_1+p_2)$ subject to $p_0+p_1 \geq \frac{5}{27}$ is $\frac{10}{27} < \frac{7}{9}$). But it inherits the same pathology as the conjecture: the Turán limit attains $\frac{5}{27}$ exactly, so a flag proof of it also requires exact tightness. It holds with equality on the 3-partite Turán blow-ups—a sub-family on which $p_1 = 0$, and therefore excluding the $k \geq 2$ members where the level-6 residual actually lives. The minimiser $(0, 0, 1, 0)$ of the linear relaxation, we note, is not realisable by any limit object; it is attainable at finite n if and only if $n \leq 6$, the $n = 6$ witness being a self-complementary 10-edge $K_4^{(3)}$-free graph with automorphism group of order 60.

Theorem 14 (Family-only certificates die at levels 6 and 7)**.** *No positive-semidefinite certificate supported on the family-realised level-*7 *profiles reaches* $\frac{5}{9}$*; the family-only certificates are certified dead at levels* 6 *and* 7*.*

This is a scoped impossibility statement about a named certificate class, and it does not transfer to the colour-refined cone, in which we hold a certificate below the family ceiling. Two further routes close by the same tightness principle.

Theorem 15 (No link ceiling)**.** $K_4^{(3)}$ *admits no link-density ceiling of the usual kind.* K_r*-free links give ceilings* $(r-2)/(r-1)$*, and* $\frac{5}{9}$ *lies strictly inside the gap* $(\frac{1}{2}, \frac{2}{3})$*, so no such ceiling lands in the interval* $(\frac{5}{9}, 0.5616)$ *that would help. The coupled ceiling* $|E| \leq \deg(v) + [\binom{n-1}{3} - t(L(v))]$ *is, on average over* v*, the tautology* $0 \leq 4p_0 + 3p_1 + 2p_2$*.*

Theorem 16 (Blow-up sweeps cannot close a row)**.** *For* $m \geq m_0$*,* $d(m) \leq \dfrac{\mathrm{ex}(m_0)}{\binom{m_0}{3}} \cdot \dfrac{m-2}{m+1}$*. Hence no finite iterated blow-up sweep can close the gap; only an initial segment can.*

What is bracketed, and what is not. The method's own value, the level-6 semidefinite optimum λ_6, is bracketed from below for the first time: $\lambda_6 \geq 107800753/199999980 = 0.539003818900$, certified. This bound is weak—it lies below $\frac{5}{9}$—so it does not decide the lane, and the value in the interval $(\frac{5}{9}, 0.5604\ldots)$ that would decide it remains unbracketed from below. A second mechanism, the Balogh–Clemen–Lidický ℓ_2 inequality $\mathbb{E}[d(u,v)^2] \leq \frac{1}{3}$, lifts to an exact slack table (Turán 0, Brown $\frac{1}{108}$, Kostochka $\frac{8}{729}$) giving $\lambda(\mu) \geq \frac{5}{9} + \frac{8\mu}{729}$, but the semidefinite program independently returned $\mu = 0$, closing it. The space of external inequalities that could in principle help—those constant on the whole family—is a 956-dimensional annihilator at $N = 6$, and it is unsearched.

Prior work. A search of nineteen full texts (Razborov's papers on the $(3,4)$-problem and his 2013 Interim Report, Keevash's survey, Baber's thesis, Baber–Talbot, and the two Falgas-Ravry–Vaughan papers, 1.65 MB in all) found no occurrence of the inequality $p_0 + p_1 \geq \frac{5}{27}$, nor of the constant $\frac{5}{27}$ in this context; the search was checked against the failure mode in which a typesetting system renders displayed fractions as stacked digits, by reading every bare "27" denominator by hand. The one printed constraint on the 4-profile body we did find is Falgas-Ravry and Vaughan's $\pi_{K_4^{(3)-}}(K_4^{(3)}) = \frac{16}{27}$ [3], i.e. $p_3 \leq \frac{16}{27}$; it is not tight on the family (the $k = 2$ member has $p_3 = \frac{61}{108} < \frac{16}{27}$).

The convergence theorem for the method—that the provable cone is dense in the semantic cone [14, 12]—guarantees that the bounds converge to $\pi(K_4^{(3)})$, but says nothing about exact tightness at any finite level; and Hatami and Norine [9] show the semantic cone is not recursively enumerable, so any proof system of this kind is necessarily incomplete. Falgas-Ravry and Vaughan predicted in print [2] that flag algebra cannot be made tight for unstable problems— $K_4^{(3)}$ has exponentially many non-isomorphic extremal configurations—and labelled this "a heuristic justification for our pessimism." It is a belief, not a theorem, and our certified family-only deaths of Theorem 14 are a finite instance in its favour.

6 Computational notes

The obstruction to level 7 is not memory and not time; the figures that suggested otherwise were artifacts. An interior-point solve of the plain level-7 program would need a dense scaling matrix of 403.3 GB and is dead, but a first-order solve needs only 22.2 MiB of cone memory and 0.71 Gflop per iteration, and a real cutting-plane solve with 1,000 active rows peaks at 3.15 GiB of resident memory; the 60.3 GB out-of-memory failure on record was the monolithic assembly, not the cutting-plane path. The real level-7 constraint matrix is 1,472,430 × 1,464,431 with 12.4M nonzeros and 133 semidefinite blocks of maximum dimension 378 and total dimension 15,527.

Two costing errors are worth publishing as warnings. First, a block-structure step that takes 2.7 s on an unloaded machine took 527 s on a loaded one, a factor of 195, and an inflation of 307× was measured on a single identical step under self-inflicted load; *timing under load measures the scheduler, not the code,* and any pricing of a flag-algebra computation must record and gate on machine load. Second, an assembly-only figure was for some time treated as the cost of *deciding* a level-7 question, although the script producing it contains no solve step at all; one must price the whole job, not the part one happened to measure.

The colour refinement is not free: plain level 6 has total block dimension 228, the coloured level 6 has 3,156, a factor of about fourteen. The coloured level-7 program has never been built.

7 Concluding remarks

The bound of this paper is the endpoint of a local descent in the colour-refined level-6 cone, not the optimum of that program; an upper bound needs no optimality, so this costs the result nothing, but it is why the method's own ceiling—call it λ_{col}, known only to lie in $[\frac{5}{9}, 0.5604007071302298]$—remains the decisive unknown. Concrete progress would take one of a few forms: bracketing λ_{col} from below within the interval that matters, where a single rational dual-feasible point above $\frac{5}{9}$ would prove the lane capped and feasibility alone suffices; building the coloured level-7 program, now known to be affordable; searching the 956-dimensional space of family-constant inequalities; or a genuinely new device capable of forcing exact tightness on an unstable extremal family, which Proposition 12 and Theorem 13 show is what the problem actually demands.

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